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雑誌名	Tsukuba journal of mathematics
巻	4
号	2
ページ	157-159
発行年	1980-12
URL	http://hdl.handle.net/2241/7136

TSUKUBA J. MATH.
Vol. 4 No. 2 (1980). 157–159

A NOTE ON THE HEREDITARY PROPERTIES IN THE PRODUCT SPACE

By

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In this note we shall investigate some hereditary properties of a subspace of a product space.

Let X_α be a topological space for each $\alpha \in I$ and A be a subset of I. p_A is the projection: $\prod_{\alpha\in I} X_\alpha \to \prod_{\alpha\in A} X_\alpha$, *i.e.* $p_A(x)$ is the restricted function of x whose domain is A. A is co-countable if $I-A$ is countable.

The family of sets is linked if each pair of its members has non-empty intersection. The space has (K)-property (precaliber $\aleph_1$) (caliber $\aleph_1$), if any uncountable family of non-empty open subsets of X includes an uncountable subfamily which is linked (has the finite intersection property) (has non-empty intersection). [2]

THEOREM. *Let X_α be second-countable for $\alpha \in I$ and X be a subspace of $\prod_{\alpha\in I} X_\alpha$ and ϕ be one of the properties:*

1) *the countable chain condition,* 2) (K)*-property,* 3) *precaliber* $\aleph_1$, 4) *caliber* $\aleph_1$, 5) *the seperability,* 6) *the Lindelöf property.*

Then, X satisfies the hereditarily ϕ if and only if for any subspace Y of X, there exists co-countable subset A of I such that $p_A{}''Y$ satisfies ϕ.

LEMMA 1. (N. A. Sanin) [1] *Let Γ be an uncountable set of finite sets, then Γ includes an uncountable subfamily Δ which is quasi-disjoint i.e. $x \cap y \subseteq \bigcap \Delta$ for each different x and y of Δ.*

See [1] for the proof.

LEMMA 2. *Let f be a continuous function whose domain is X and X satisfies ϕ in the theorem. Then, the range of f also satisfies ϕ.*

Proof. Easy to check.

Let $\{V_n^\alpha;\ n<\omega\}$ be a base of X_α. Then, $\{p_A^{-1} V_{n_1}^{\alpha_1} \times \cdots \times V_{n_m}^{\alpha_m};\ A=\{\alpha_1, \cdots, \alpha_m\}, m<\omega\} = \mathcal{V}$ is a base of $\prod_{\alpha\in I} X_\alpha$. The domain of the basic open set $V (= p_A^{-1} V_{n_1}^{\alpha_1} \times$

Received November 19, 1979
This research was partially supported by Grand-in-Aid for Co-operativ Research, Project No. 434007.

$\cdots \times V_{n_m}^{\alpha_m})$ is A, which is finite, and is denoted by dom V.

LEMMA 3. *Let θ be an unconntable subfamily of $\mathcal{V}$. Then, θ includes an uncountable subfamily Φ which has the following properties:*

a) $\{\text{dom } V;\ V \in \Phi\}$ *is quasi-disjoint,*

b) $p_A'' V = p_A'' W$ *for each* $V, W \in \Phi$, *where* $A = \bigcap \{\text{dom } V;\ V \in \Phi\}$.

PROOF. By Lemma 1, θ includes an uncountable subfamily θ' such that $\{\text{dom } V;\ V \in \theta'\}$ is quasi-disjoint.

Let $A = \bigcap \{\text{dom } V;\ V \in \theta'\}$. Then, $\{p_A'' V;\ V \in \theta'\}$ is countable. Hence, some uncountable subfamily Φ of θ' has the properties in the lemma.

PROOF OF THEOREM. The necessity is clear and so we shall prove the sufficiency. Suppose that X does not satisfy the hereditarily ϕ. Then, there exists a subset $\{x_\alpha;\ \alpha < \omega_1\}$ and a family $\{O_\alpha;\ \alpha < \omega_1\}$ of open subsets of X such that $x_\alpha \in O_\alpha$ for each $\alpha < \omega_1$ and

i) $x_\alpha \notin O_\beta$ for any $\beta \neq \alpha$,

ii) for any uncountable subset S of ω_1, there exists a pair α, β of S; $O_\alpha \cap O_\beta \cap \{x_\alpha;\ \alpha < \omega_1\} = \phi$,

iii) for any uncountable subset S of ω_1, there exists a finite subset F of S; $\bigcap_{\alpha \in F} O_\alpha \cap \{x_\alpha;\ \alpha < \omega_1\} = \phi$,

iv) for any uncountable subset S of ω_1, $\bigcap_{\alpha \in S} O_\alpha \cap \{x_\alpha;\ \alpha < \omega_1\} = \phi$,

v) $x_\alpha \notin O_\beta$ for any $\beta > \alpha$, or

vi) $x_\alpha \notin O_\beta$ for any $\beta < \alpha$, according to ϕ is 1), 2), 3), 4), 5) or 6), respectively.

We may take the above $O_\alpha (\alpha < \omega_1)$ from $\mathcal{V}$. By Lemma 3, without a loss of generality we can assume that $\{O_\alpha;\ \alpha < \omega_1\}$ satisfies the conditions a) and b) of Lemma 3.

Now, we apply the assumption and Lemma 2 to $\{x_\alpha;\ \alpha < \omega_1\}$. Then, there exists a co-countable subset A of I such that $p_A'' \{x_\alpha;\ \alpha < \omega_1\}$ satisfies ϕ and $\bigcap \{\text{dom } O_\alpha;\ \alpha < \omega_1\} \cap A$ is empty. Since $\{\text{dom } O_\alpha;\ \alpha < \omega_1\}$ is quasi-disjoint, we may assume $\text{dom } O_\alpha - \bigcap \{\text{dom } O_\alpha;\ \alpha < \omega_1\} \subseteq A$ for $\alpha < \omega_1$. There exists(s)

i)' α such that $p_A(x_\alpha) \in p_A'' O_\beta \cap p_A'' O_\gamma$ for some $\beta \neq \gamma$,

ii)' an uncountable subset S of ω_1 such that $p_A'' O_\alpha \cap p_A'' O_\beta \cap p_A'' \{x_\alpha;\ \alpha < \omega_1\} \neq \phi$ for each distinct $\alpha, \beta \in S$,

iii)' an uncountable subset S of ω_1 such that $\bigcap_{\alpha \in F} p_A'' O_\alpha \cap p_A'' \{x_\alpha;\ \alpha < \omega_1\} \neq \phi$ for any finite $F \subseteq S$, iv)′ α and an uncountable subset S of ω_1 such that $p_A(x_\alpha) \in \bigcap_{\alpha \in S} p''_A O_\alpha$,

v)′ α such that $p_A(x_\alpha) \in p''_A O_\beta$ for some $\beta > \alpha$, or

vi)′ α such that $p_A(x_\alpha) \in p''_A O_\beta$ for some $\beta < \alpha$,

according that ψ is 1), 2), 3), 4), 5) or 6), respectively.

By the assumption of A and the fact $x_\alpha \in O_\alpha$, $p_A(x_\alpha) \in p''_A O_\beta$ holds if and only if $x_\alpha \in O_\beta$ holds, for each α, β. So, i)′, ⋯, or vi)′ contradicts to i), ⋯, or vi) respectively.

Now, the proof is complete.

Since the hereditary separability is equivalent to the hereditary caliber-$\aleph_1$-property, it is a little interesting to compare the two cases 4) and 5) in the theorem.

References

[1] Juhasz, I., Cardinal Functions in Topology, Amsterdam (1971).
[2] Kunen, K. and Tall, F.D., Between Martin's Axiom and Souslin's Hypothesis, Fundamenta Mathematicae CII. 3 (1979).

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